Quiz #7 Solutions

1. (a) By the Rank Theorem, $\dim \operatorname{Nul} A = n - \operatorname{rank} = 4 - 2 = 2$.

(b) All four given vectors are solutions of $A\mathbf{x} = \mathbf{0}$ and, so, are vectors in $\operatorname{Nul} A$. By the Basis Theorem, any linearly independent set of two of these vectors will form a basis for $\operatorname{Nul} A$. For example, $\{\mathbf{p}_1, \mathbf{p}_2\}$ forms a basis for $\operatorname{Nul} A$ because neither vector is a scalar multiple of the other, so they are linearly independent. (Actually, in this example, *any two* of the four vectors will work as a basis.)

2. (a) The coefficient matrix is

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \end{bmatrix}$$

The augmented matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 \end{bmatrix}$$

is already in reduced echelon form, and its general solution is given by

$$\begin{cases} x_1 = -x_3 - x_4 - 2x_5 \\ x_2 = -x_4 - 3x_5 \\ x_3,\ x_4,\ x_5 \text{ free} \end{cases}$$

giving a vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 - 2x_5 \\ -x_4 - 3x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3,\ x_4,\ x_5 \text{ free}$$

This gives a basis for $\operatorname{Nul} A$ of:

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

which gives $\dim \operatorname{Nul} A = 3$.

(b) The new coefficient matrix B is given by

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 2 & -1 & 2 & 2 & 1 \end{bmatrix}$$

Solving the homogeneous equation, we get:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 2 & -1 & 2 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R1} \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & -1 & 0 & 0 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + R2} \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[R2 \to R2 - R3]{R1 \to R1 - R3} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

which gives the general solution:

$$\begin{cases} x_1 = -x_3 - 2x_5 \\ x_2 = -3x_5 \\ x_4 = 0 \\ x_3,\ x_5 \text{ free} \end{cases}$$

and vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - 2x_5 \\ -3x_5 \\ x_3 \\ 0 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3,\ x_5 \text{ free}$$

This gives a basis for $\operatorname{Nul} B$ of:

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

which gives $\dim \operatorname{Nul} B = 2$.

3. (a) The determinant of a triangular matrix is the product of its diagonal elements:

$$\det A = (1)(2)(3)(4) = 24$$

(b) We calculate the inverse of A in the usual manner:

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{2} & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R4 \to \frac{1}{4} R4} \begin{bmatrix} \boxed{1} & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{2} & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$
$$\qquad\quad\Uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Uparrow$$

$$\xrightarrow{R1\to R1-2R4}\begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & \boxed{2} & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \xrightarrow{R2\to R2-R4}\begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & \boxed{2} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\xrightarrow{R3\to \frac{1}{3}R3}\begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & \boxed{2} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \xrightarrow{R1\to R1-R3}\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{2} \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\xrightarrow{R2\to \frac{1}{2}R2}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

This gives the inverse

$$A^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{8} \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

(c) The eigenvalues of a triangular matrix are its diagonal entries, so the eigenvalues of A are 1, 2, 3, and 4, and the eigenvalues of A^{-1} (also a triangular matrix) are 1, 1/2, 1/3, and 1/4.

(d) A's smallest eigenvalue is 1. It's eigenspace is $\mathrm{Nul}(A-I)$. We have

$$A - I = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and we find the null space of this matrix by the following (tricky) row reduction of the augmented matrix

$$\begin{bmatrix} 0&0&1&2&0\\ 0&1&0&1&0\\ 0&0&2&0&0\\ 0&0&0&3&0 \end{bmatrix} \xrightarrow{R1\leftrightarrow R2} \begin{bmatrix} 0&1&0&1&0\\ 0&0&1&2&0\\ 0&0&2&0&0\\ 0&0&0&3&0 \end{bmatrix}$$

$$\xrightarrow{R3\rightarrow R3-2R2} \begin{bmatrix} 0&1&0&1&0\\ 0&0&1&2&0\\ 0&0&0&-4&0\\ 0&0&0&3&0 \end{bmatrix} \xrightarrow{R4\rightarrow R4+\frac{3}{4}R3} \begin{bmatrix} 0&\boxed{1}&0&1&0\\ 0&0&\boxed{1}&2&0\\ 0&0&0&-4&0\\ 0&0&0&0&0 \end{bmatrix}$$

$$\xrightarrow{R3\rightarrow -\frac{1}{4}R3} \begin{bmatrix} 0&\boxed{1}&0&1&0\\ 0&0&\boxed{1}&2&0\\ 0&0&0&1&0\\ 0&0&0&0&0 \end{bmatrix} \xrightarrow{R1\rightarrow R1-R3} \begin{bmatrix} 0&\boxed{1}&0&0&0\\ 0&0&\boxed{1}&2&0\\ 0&0&0&1&0\\ 0&0&0&0&0 \end{bmatrix}$$

$$\xrightarrow{R2\rightarrow R2-2R3} \begin{bmatrix} 0&\boxed{1}&0&0&0\\ 0&0&\boxed{1}&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0 \end{bmatrix}$$

This gives the (tricky) general solution

$$\begin{cases} x_2 = 0\\ x_3 = 0\\ x_4 = 0\\ x_1 \text{ free} \end{cases}$$

and its (tricky) vector parametric form

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} = \begin{bmatrix} x_1\\ 0\\ 0\\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1\\ 0\\ 0\\ 0 \end{bmatrix}, \quad x_1 \text{ free}$$

Thus the eigenspace associated with the eigenvalue 1 has basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

4. (a) Note that the characteristic equation has the form:

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = -\lambda(3-\lambda) - (-2) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

So, the characteristic equation has solutions $\lambda = 1$ and $\lambda = 2$, and these are the two eigenvalues of A.

(b) For the eigenvalue $\lambda = 1$, we find the null space of $A - I$:

$$\begin{bmatrix} -1 & -2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + R1} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to -R1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives general solution

$$\begin{cases} x_1 = -2x_2 \\ x_2 \text{ free} \end{cases}$$

and parametric vector form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad x_2 \text{ free}$$

so a basis for this eigenspace is $\{(-2, 1)\}$.
For the eigenvalue $\lambda = 2$, we find the null space of $A - 2I$:

$$\begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + 2R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives general solution

$$\begin{cases} x_1 = -x_2 \\ x_2 \text{ free} \end{cases}$$

and parametric vector form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad x_2 \text{ free}$$

so a basis for this eigenspace is $\{(-1, 1)\}$.

(c) The matrix P has columns given by the basis vectors calculated in part (b). That is,

$$P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

and D is the diagonal matrix with the corresponding eigenvalues

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

(d) By the inverse formula for 2×2 matrices:

$$P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

Checking:

$$PDP^{-1} = \left(\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} = A$$

(e)

$$\begin{aligned} A^{10} = PD^{10}P^{-1} &= \left(\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}\right)\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -1024 \\ 1 & 1024 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1022 & -2046 \\ 1023 & 2047 \end{bmatrix} \end{aligned}$$

5. (a) Note that

$$AB\mathbf{v} = A(B\mathbf{v}) = A(3\mathbf{v}) = 3(A\mathbf{v}) = 3(2\mathbf{v}) = 6\mathbf{v}$$

By definition, $\mathbf{v}$ is an eigenvector of AB with eigenvalue 6.

(b) Note that

$$\det(C - 2I) = \det\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

so 2 is an eigenvalue of C. Similarly,

$$\det(D - 3I) = \det\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} = 0$$

so 3 is an eigenvalue of D. However, since

$$\det(CD - 6I) = \det\left(\begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right) = \det\begin{bmatrix} -4 & 3 \\ 2 & 0 \end{bmatrix} = -6 \neq 0$$

it follows that 6 is *not* an eigenvalue of CD.