Variable Substitution

"Say, didn't he say he was going to show us how to calculate that integral at the end of class?"

Okay, I forgot, but here's a rapid review of the integration technique known as "variable substitution" or "change of variables."

Variable Substitution

Say we have an integral we want to evaluate, like:

$$\int_a^b f(t)dt \tag{1}$$

If we write $t = h(u)$ as a function of another variable u, then under some conditions (f continuous, h having continuous derivative), it's true that

$$\int_a^b f(t)dt = \int_\alpha^\beta f(h(u))h'(u)du \tag{2}$$

where α and β are chosen so that $a = h(\alpha)$ and $b = h(\beta)$. If we pick the right $t = h(u)$, the new integral (2) will be much easier to evaluate than the old one (1).

Example from Class

Take the example from class. We want to evaluate the integral:

$$\int_0^T \sqrt{1 + \tfrac{9}{4}t}\, dt = \int_0^T \left(1 + \tfrac{9}{4}t\right)^{1/2} dt$$

Note that if $u = 1 + \frac{9}{4}t$, so that $t = h(u) = \frac{4}{9}(u-1)$, then $h'(u) = \frac{4}{9}$, so we can write:

$$\begin{aligned}
\int_0^T \left(1 + \tfrac{9}{4}t\right)^{1/2} dt &= \int_\alpha^\beta \left(1 + \tfrac{9}{4}h(u)\right)^{1/2} h'(u)du \\
&= \int_1^{1+\frac{9}{4}T} u^{1/2}\tfrac{4}{9}\, du \\
&= \frac{4}{9}\frac{u^{3/2}}{3/2}\Bigg|_1^{1+\frac{9}{4}T} \\
&= \tfrac{8}{27}(1 + \tfrac{9}{4}T)^{3/2} - \tfrac{8}{27}
\end{aligned}$$

Because $8 = 4^{3/2}$, with a little manipulation we get the same answer as in class:

$$\frac{(4 + 9T)^{3/2}}{27} - \frac{8}{27}$$

Note that actually applying formula (2) is a little gross. Most people, including me, play a little fast and loose and do it this way. To evaluate the integral:

$$\int_0^T \left(1 + \tfrac{9}{4}t\right)^{1/2} dt$$

I know I'd like to make the variable substitution $u = 1 + \frac{9}{4}t$ so I'll be taking the integral of the square root of a single variable (something we have a rule for) instead of a more complex expression. Because $u = 1 + \frac{9}{4}t$, we can write

$$\frac{du}{dt} = \frac{9}{4}$$

and so (this is the fast and loose part), we can pretend for a minute that the du and dt can be broken apart and write $du = \frac{9}{4}dt$ and so $dt = \frac{4}{9}du$.

Now, it's just a matter of rewriting all the parts that involve t (including the limits of integration) in terms of u:

$$\int_{t=0}^{t=T} \left(1 + \tfrac{9}{4}t\right)^{1/2} dt = \int_{u=1+\frac{9}{4}0}^{u=1+\frac{9}{4}T} u^{1/2}\tfrac{4}{9}\, du$$

and the rest of the calculation proceeds as above.

A Trigonometric Substitution

A slightly more interesting example is the following integral:

$$\int_a^b \frac{1}{1+x^2} dx$$

This is one of a number of integrals where substituting x with a trigonometric function of another variable u is helpful. Knowing which substitution to use is a matter of trial and error (or memorization). Here, note that the substitution $x = \tan u$ (and so $u = \tan^{-1} x$) has derivative $dx = \sec^2 u\, du$, and so:

$$\int_{x=a}^{x=b} \frac{1}{1+x^2} dx = \int_{u=\tan^{-1} a}^{u=\tan^{-1} b} \frac{1}{1+\tan^2 u} \sec^2 u\, du$$

The reason this is an easier integral to work with is because of the trigonometric identity $1 + \tan^2 u = \sec^2 u$ (which you can derive by dividing both sides of $\sin^2 u + \cos^2 u = 1$ by $\cos^2 u$). As a result, we have:

$$\int_{x=a}^{x=b} \frac{1}{1+x^2} dx = \int_{u=\tan^{-1} a}^{u=\tan^{-1} b} 1\, du = u\Big|_{u=\tan^{-1} a}^{u=\tan^{-1} b} = \tan^{-1} b - \tan^{-1} a$$

If you look at the table of integrals at the back of your textbook, near the bottom of the "Elementary Integrals" (ha!) section, you'll see the formula:

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

which for $a = 1$ gives:

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

or, for a definite integral:

$$\int_a^b \frac{dx}{1+x^2} = \tan^{-1} b - \tan^{-1} a$$

which is what we calculated with our substitution.